Physics 319
Classical Mechanics

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Lecture 15

Two Body Problem

- Consider the motion of two bodies with mass m_1 and m_2, interacting through an interaction potential U

$$\mathcal{L} = m_1 \frac{\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_1}{2} + m_2 \frac{\dot{\vec{r}}_2 \cdot \dot{\vec{r}}_2}{2} - U\left(\vec{r}_1 - \vec{r}_2\right)$$

- Euler-Lagrange equations are the correct

$$m_1 \frac{d^2 \vec{r}_1}{dt^2} = -\frac{\partial U}{\partial \vec{r}_1} = \vec{F}_{12}$$

$$m_2 \frac{d^2 \vec{r}_2}{dt^2} = -\frac{\partial U}{\partial \vec{r}_2} = \vec{F}_{21} = -\vec{F}_{12}$$

- Center of Mass motion trivial

$$\frac{d^2 \vec{r}_{cm}}{dt^2} = \vec{F}_{12} + \vec{F}_{21} = 0 \rightarrow \vec{v}_{cm} = const$$

Relative Co-ordinates

- Recall from Lecture 5

$$\vec{r}_1 = \vec{r}_{cm} + \frac{m_2}{m_1 + m_2}\left(\vec{r}_1 - \vec{r}_2\right)$$

$$\vec{r}_2 = \vec{r}_{cm} - \frac{m_1}{m_1 + m_2}\left(\vec{r}_1 - \vec{r}_2\right)$$

$$\dot{\vec{r}}_1 = \dot{\vec{r}}_{cm} + \frac{m_2}{m_1 + m_2}\left(\dot{\vec{r}}_1 - \dot{\vec{r}}_2\right)$$

$$\dot{\vec{r}}_2 = \dot{\vec{r}}_{cm} - \frac{m_1}{m_1 + m_2}\left(\dot{\vec{r}}_1 - \dot{\vec{r}}_2\right)$$

- Lagrangian in new co-ordinates

$$\mathcal{L} = \left(m_1 + m_2\right)\frac{\dot{\vec{r}}_{cm} \cdot \dot{\vec{r}}_{cm}}{2} + \mu\frac{\left(\dot{\vec{r}}_1 - \dot{\vec{r}}_2\right)\cdot\left(\dot{\vec{r}}_1 - \dot{\vec{r}}_2\right)}{2} - U\left(\vec{r}_1 - \vec{r}_2\right) \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

- μ is called the reduced mass

Reduced to One-body problem

- In terms of relative co-ordinate

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\mathcal{L}\left(\dot{\vec{r}}_{cm}, \dot{\vec{r}}, \vec{r}\right) = \left(m_1 + m_2\right)\frac{\dot{\vec{r}}_{cm}\cdot\dot{\vec{r}}_{cm}}{2} + \mu\frac{\dot{\vec{r}}\cdot\dot{\vec{r}}}{2} - U\left(\vec{r}\right)$$

- Euler-Lagrange for CM motion reproduces momentum conservation argument. Go into frame where the CM is at rest. The remaining problem is a three degree of freedom one with Lagrangian

$$\mathcal{L} = \mathcal{L}_{rel}\left(\dot{\vec{r}}, \vec{r}\right) = \mu\frac{\dot{\vec{r}}\cdot\dot{\vec{r}}}{2} - U\left(\vec{r}\right)$$

- For a central force

$$U\left(\vec{r}\right) = U\left(r\right)$$

$$\therefore\ \mu\ddot{\vec{r}} = -\frac{\partial U}{\partial r}\hat{r}$$

Angular Momentum Conserved

- Total angular momentum vector

$$\vec{L} = m_1\vec{r}_1 \times \dot{\vec{r}}_1 + m_2\vec{r}_2 \times \dot{\vec{r}}_2$$

$$= \frac{m_1 m_2^2}{\left(m_1 + m_2\right)^2}\vec{r} \times \dot{\vec{r}} + \frac{m_2 m_1^2}{\left(m_1 + m_2\right)^2}\left(-\vec{r}\right) \times \left(-\dot{\vec{r}}\right)$$

$$= \mu\vec{r} \times \dot{\vec{r}}$$

is conserved

$$\dot{\vec{L}} = \mu\dot{\vec{r}} \times \dot{\vec{r}} + \mu\vec{r} \times \ddot{\vec{r}} = 0 - \frac{\partial U}{\partial r} r\hat{r} \times \hat{r} = 0$$

- Thus motion in a plane. Put polar coordinate system in the plane of the orbit. Problem is

$$\mathcal{L} = \mu\frac{\dot{r}^2 + r^2\dot{\theta}^2}{2} - U\left(r\right)$$

Reduction to one dimension

- The magnitude of the conserved angular momentum is

$$l = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \mu r^2 \dot{\theta}$$

- Radial equation of motion

$$\mu \ddot{r} = \mu r \dot{\theta}^2 - \frac{\partial U}{\partial r} = \frac{l^2}{\mu r^3} - \frac{\partial U}{\partial r} = -\frac{\partial}{\partial r}\left[U + \frac{l^2}{2\mu r^2}\right]$$

- Effective one dimensional potential

$$U_{eff}\left(r\right) = U\left(r\right) + \frac{l^2}{2\mu r^2}$$

- Newton gravity case

$$U_{eff}\left(r\right) = -\frac{Gm_1 m_2}{r} + \frac{l^2}{2\mu r^2}$$

Effective Potential

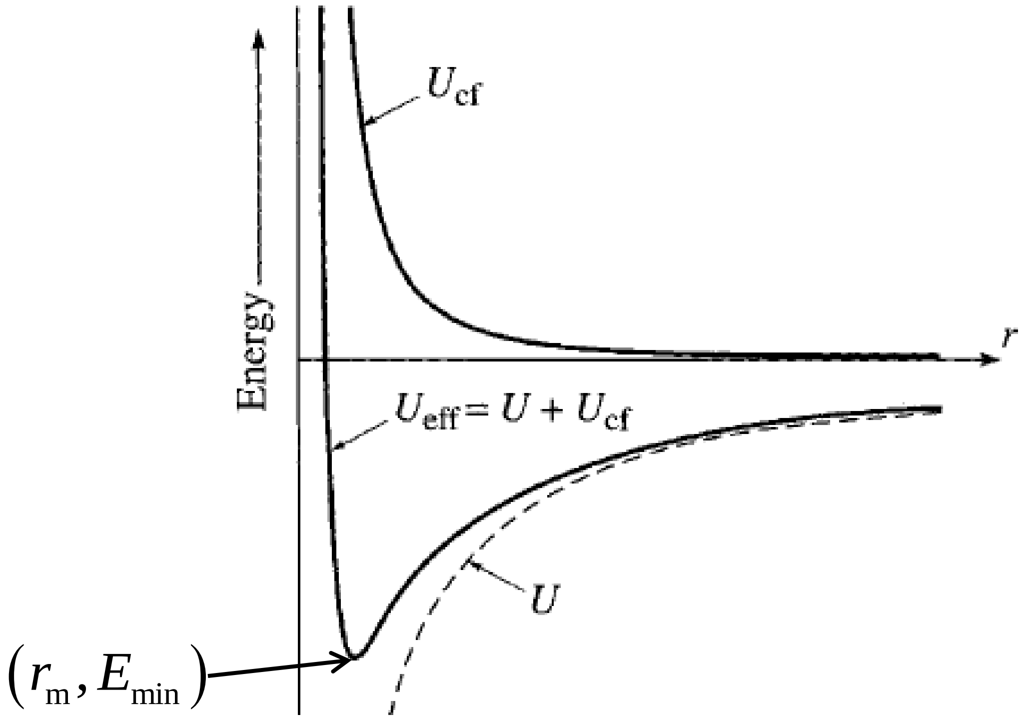


Figure 8.4 The effective potential energy $U_{eff}(r)$ that governs the radial motion of a comet is the sum of the actual gravitational potential energy $U(r) = -Gm_1m_2/r$ and the centrifugal term $U_{cf} = \ell^2/2\mu r^2$. For large r, the dominant effect is the attractive gravitational force; for small r, it is the repulsive centrifugal force.

Energy Constant

$$\mu \dot{r}\ddot{r} = -\dot{r}\frac{\partial}{\partial r}\left[U + \frac{l^2}{2\mu r^2}\right]$$

$$\mu\frac{\dot{r}^2}{2} + U_{eff}\left(r\right) = E = const$$

- Orbits, qualitatively
  - $E > 0$, motion is unbound
  - $E < 0$, motion is bounded between two turning points
  - $E = E_{\mathrm{min}}$, orbit a circle (no radial motion)
  - $E = 0$, motion unbound parabola

$$\frac{\partial}{\partial r}\left[U + \frac{l^2}{2\mu r^2}\right]\Bigg|_{r=r_{\mathrm{m}}} = 0 \rightarrow r_{\mathrm{m}} = \frac{l^2}{Gm_1 m_2 \mu}$$

Taylor's *c*

$$E_{\mathrm{min}} = -\frac{\left(Gm_1 m_2\right)^2 \mu}{l^2} + \frac{\left(Gm_1 m_2\right)^2 \mu}{2l^2} = -\frac{\left(Gm_1 m_2\right)^2 \mu}{2l^2}$$

Elliptical Motion

- First determine the constant Laplace-Runge-Lenz vector

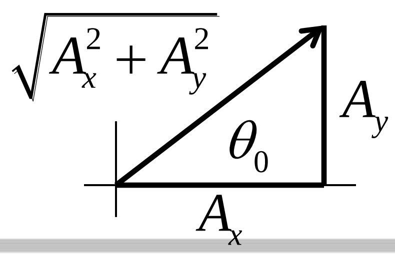
$$\frac{d}{dt}\cos\theta = -\dot{\theta}\sin\theta = -\frac{l}{\mu r^2}\frac{y}{r} = \frac{l}{Gm_1m_2}\frac{d^2}{dt^2}r\sin\theta$$

$$\frac{d}{dt}\sin\theta = \dot{\theta}\cos\theta = \frac{l}{\mu r^2}\frac{x}{r} = -\frac{l}{Gm_1m_2}\frac{d^2}{dt^2}r\cos\theta$$

$$-A_x = \cos\theta - \frac{l}{Gm_1m_2}\frac{d}{dt}\left(r\sin\theta\right) = \cos\theta - \frac{l}{Gm_1m_2}\left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)$$

$$-A_y = \sin\theta + \frac{l}{Gm_1m_2}\frac{d}{dt}\left(r\cos\theta\right) = \sin\theta + \frac{l}{Gm_1m_2}\left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)$$

$$A_x\hat{x} + A_y\hat{y} = \sqrt{A_x^2 + A_y^2}\cos\theta_0\hat{x} + \sqrt{A_x^2 + A_y^2}\sin\theta_0\hat{y}$$



Orbit as function of θ

- Determine the vector magnitude

$$A_x^2 + A_y^2 = 1 - 2\frac{l}{Gm_1m_2} r\dot{\theta} + \frac{l^2}{(Gm_1m_2)^2} r^2\dot{\theta}^2 + \frac{l^2}{(Gm_1m_2)^2}\dot{r}^2 = 1 + \frac{2El^2}{(Gm_1m_2)^2\mu}$$

- Equation for radius $r(\theta)$

$$\vec{L} = l\hat{z} = \mu r^2\dot{\theta}\hat{z}$$

$$r\cos\theta A_x + r\sin\theta A_y = -r + \frac{l^2}{Gm_1m_2\mu}$$

$$= r\sqrt{A_x^2 + A_y^2}\cos(\theta - \theta_0)$$

$$\therefore r(\theta) = \frac{\left(l^2/(Gm_1m_2\mu)\right)}{1 + \sqrt{1 + \frac{2El^2}{(Gm_1m_2)^2\mu}}\cos(\theta - \theta_0)} = \frac{r_{\mathrm{m}}}{1 + \sqrt{1 - (E/E_{\mathrm{min}})}\cos(\theta - \theta_0)}$$